S161. Let ABC be a triangle inscribed in a circle of center O and radius R. If d_A, d_B, d_C are the distances from O to the sides of the triangle, prove that

$$R^3 - (d_A^2 + d_B^2 + d_C^2)R - 2d_Ad_Bd_C = 0.$$

Proposed by Dorin Andrica, Babes-Bolyai University, Cluj-Napoca, Romania

First solution by Arkady Alt, San Jose, California, USA

Since $\triangle COB$ is isosceles and$\angle COB = 2\widehat{A}$ then $\frac{d_a}{R} = \cos A$, and cyclicly $\frac{d_b}{R} = \cos B, \frac{d_c}{R} = \cos C$. By substitution in the well known identity $\cos^2 A + \cos^2 B + \cos^2 C + 2\cos A \cos B \cos C = 1$ we obtain

$$\frac{d_a^2}{R^2} + \frac{d_b^2}{R^2} + \frac{d_c^2}{R^2} + \frac{2d_ad_bd_c}{R^3} = 1 \iff R^3 - \left(d_a^2 + d_b^2 + d_c^2\right)R - 2d_ad_bd_c = 0.$$

Since the function $\varphi(x) \ := 1 - \frac{2d_ad_bd_c}{x^3} - \frac{d_a^2 + d_b^2 + d_c^2}{x^2}$ is increasing on $(0, \infty)$ then R is a single positive root of the cubic equation $x^3 - x\left(k^2 + l^2 + m^2\right) - 2klm = 0.$

Remark. This problem is part of the problem #11443 published in **The American Mathematical Monthly, Vol.116, N.6, June-July 2009.**

Problem 11443. Proposed by Eugen Ionascu, Columbus state University, Columbus, GA.

Consider a triangle ABC with circumcenter O and circumradius R.Denote the distances from O to the sides AB, BC, CA,respectively, by x, y, z. Prove that if ABC is acute then $R^3 - \left(x^2 + y^2 + z^2\right)R = 2xyz$,and $\left(x^2 + y^2 + z^2\right)R - R^3 = 2xyz$ otherwise.

A stronger statement, namely

Let x, y, z be arbitrary real positive numbers, then there exist a unique acute triangle with sidelenghts $a = 2\sqrt{R^2 - x^2}, b = 2\sqrt{R^2 - y^2}, c = 2\sqrt{R^2 - z^2}$, where R (circumradius) is the only positive root of a cubic equation $t^3 - t\left(x^2 + y^2 + z^2\right) - 2xyz = 0$, and for which numbers x, y, z be the distances from a circumcenter to the sides of this triangle

is **part i**. of the **Theorem 1** in **[1].**

[1] Arkady Alt, An independent parametrization of an acute triangle and its applications,

Mathematical Reflections 2009, Issue 4.

Second solution by Sayan Mukherjee, Kolkata, India

Note that we have, $\frac{d_A}{OC} = \sin\angle BOC = \sin\left(90^\circ - \frac{1}{2}\angle BOC\right) = \cos\alpha$. Then

$$d_A = R\cos\alpha, d_B = R\cos\beta, d_C = R\cos\gamma,$$

which, in turn, gives us

$$\begin{aligned} R^3 - (d_A^2 + d_B^2 + d_C^2)R - 2dAdBdC &= R^3\left[1 - \cos^2\alpha - \cos^2\beta - \cos^2\gamma - 2\cos\alpha\cos\beta\cos\gamma\right] \\ &= 0; \end{aligned}$$